Tetrahedron-Intersecting Families of 3-uniform Hypergraphs

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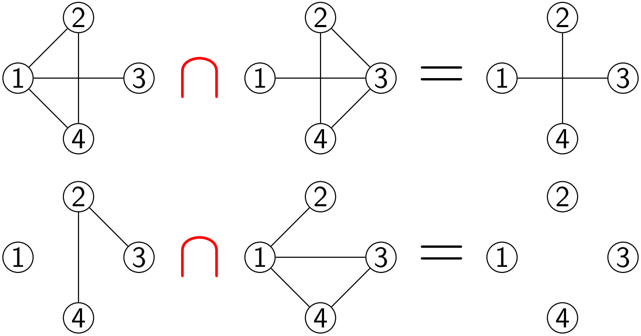
Graph Intersections

Definition

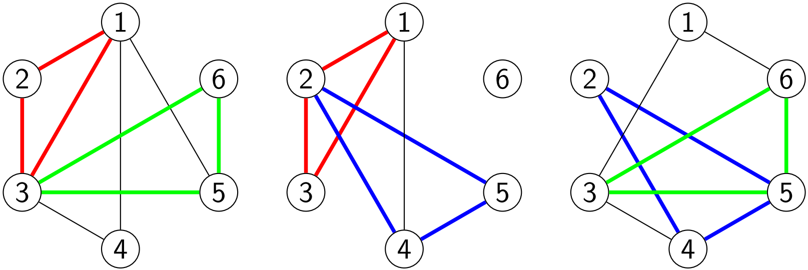
The intersection of two graphs on the same vertex set is the graph formed by the edges common to both graphs.



Triangle-Intersecting Families

Definition

A family of graphs is triangle-intersecting if the intersection of any two graphs contains some triangle as a subgraph.



Triangle-Intersecting Families

Extremal Combinatorics

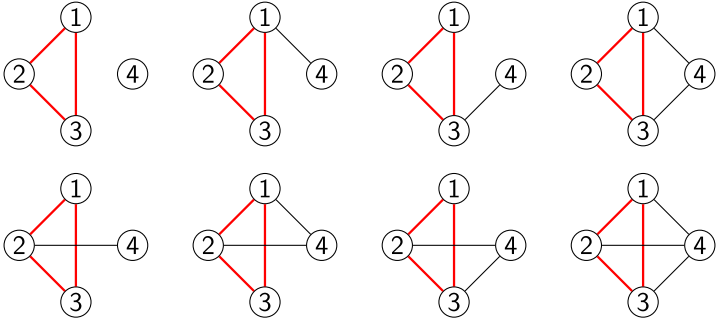
Whats the maximal size of a triangle-intersecting family of graphs on n labeled vertices?

Can we show a lower bound?

Triangle-Intersecting Families: Lower Bound

Lower Bound

At least $2^{\binom{n}{2}-3}$. Fix a triangle on the n labeled vertices, and take all graphs containing that triangle.



Triangle-Intersecting Families

Conjecture (Simonovits-Sós, 1976)

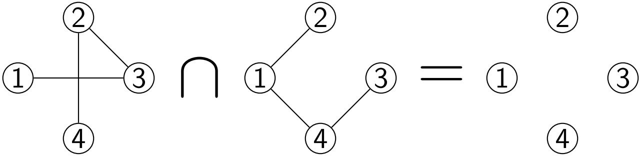
The maximal size of a triangle-intersecting family of graphs on n labeled vertices is $2^{\binom{n}{2}-3}$.

What upper bounds can we show?

Triangle-Intersecting Families: Upper Bound

Easy upper bound

Upper bound: $2^{\binom{n}{2}-1}$.
Proof. If a graph is in the family, then its complement cannot be in the family, since their intersection is empty. Hence, at most half of the graphs can be in the family.



Past Results

Theorem (Chung-Graham-Frankl-Shearer, 1986)

The maximal size of a triangle-intersecting family of graphs on n labeled vertices is $2^{\binom{n}{2}-2}$.

Proof introduced Shearer's entropy lemma, a powerful bound on the entropy of a random variable.

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Theorem (Ellis-Filmus-Friedgut, 2012)

The maximal size of a triangle-intersecting family of graphs on n labeled vertices is $2^{\binom{n}{2}-3}$.

Proved using a spectral method.

Generalizations

The natural generalization of triangle-intersecting families is

Conjecture

Given an unlabeled graph H, the maximal size of a H-intersecting family of graphs on n labeled vertices is $2^{\binom{n}{2}-e(H)}$.

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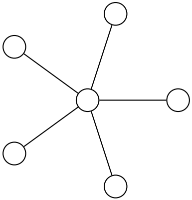
It turns out this conjecture is false; there exists an explicit counterexample when H is a star.

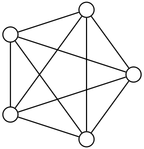


Generalizations

We focus on complete graphs rather than arbitrary graphs.

Definition

The complete graph on t vertices is denoted K_t.



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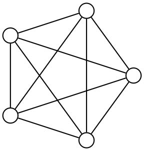


Conjecture

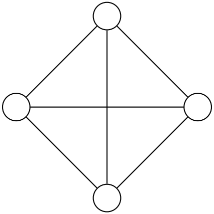
Given n and t, the maximal size of a K_t-intersecting family of graphs on n labeled vertices is $2^{\binom{n}{2}-\binom{t}{2}}$.

Past Results

The case of $t = 4$ has been proved:

Theorem (Berger-Zhao, 2021)

The maximal size of a K_4-intersecting family of graphs on n labeled vertices is $2^{\binom{n}{2}-\binom{4}{2}}$.

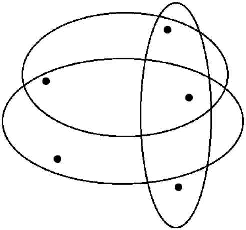


Further Generalizations

The conjecture on K_t-intersecting families of graphs can be extended further to 3-uniform hypergraphs.

Definition

A hypergraph is a generalization of a graph where an edge can join any number of vertices. A 3-uniform hypergraph is a hypergraph where each edge joins exactly three vertices.



Further Generalizations

Definition

The complete 3-uniform hypergraph on t vertices is denoted $K_t^{(3)}$.

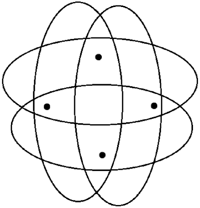
Conjecture

The maximal size of a $K_t^{(3)}$-intersecting family of 3-uniform hypergraphs on n labeled vertices is $2^{\binom{n}{3}-\binom{t}{3}}$.

Main Results

Definition

The complete 3-uniform hypergraph on 4 vertices, $K_4^{(3)}$, is called a tetrahedron.



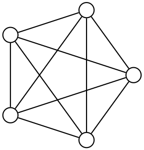
Theorem

The maximal size of a tetrahedron-intersecting family of 3-uniform hypergraphs on n labeled vertices is $2^{\binom{n}{3}-\binom{4}{3}}$.

Main Results

Theorem

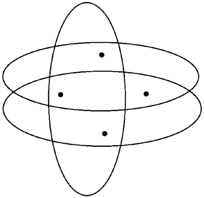
The maximal size of a K_5-intersecting family of graphs on n labeled vertices is $2^{\binom{n}{2}-\binom{5}{2}}$.



Further Directions

Definition

$K_4^{(3)-}$ is a tetrahedron with one edge removed.



- What is the maximal size of a $K_4^{(3)-}$-intersecting family?
- What are upper bounds on the size of a K_t-intersecting family for general t?

Acknowledgements

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- Thanks to the MIT PRIMES program for making this opportunity possible.

References

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[2] F.R.K Chung et al. "Some intersection theorems for ordered sets and graphs". In: *Journal of Combinatorial Theory, Series A* 43.1 (1986), pp. 23–37. ISSN: 0097-3165. DOI: https://doi.org/10.1016/0097-3165(86)90019-1. URL: https://www.sciencedirect.com/science/article/pii/0097316586900191.

[3] David Ellis, Yuval Filmus, and Ehud Friedgut. *Triangle-Intersecting Families of Graphs*. 2012. arXiv: 1010.4909 [math.CO].